Introduction to Differential Equations.

Fei Qi

Office Hr: 9:50 am – 10:50 am MTW Hill 525

Website: http://math.rutgers.edu/~fq15

Email: cl.volkov@rutgers.edu.

Differential Equation: Equation concerning derivatives.

Typical first order ordinary differential equation (ODE)

$$y' = f(t,y) \qquad y'(t) = f(t, y(t))$$

- t variable
- $y = y(t)$ one variable function depends on t
- $y' = \frac{dy}{dt}$ derivative of y
- $f(t,y)$ known function of two variables

This ODE expresses y' in terms of t, $y(t)$

General form of a first order ODE

$$F(t, y, y') = 0$$

General form of an n-th order ODE

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

Examples:

$$y' = t^2 + y$$

$$(y')^2 + \sin y = y + t^2$$

$$y''' + 3y'' + 3y' + y = e^{2t} \qquad \text{3rd order}$$

Why study ODEs?

ODE is widely used in real life.

Example: Newton's law of cooling:

Object placed in a room.

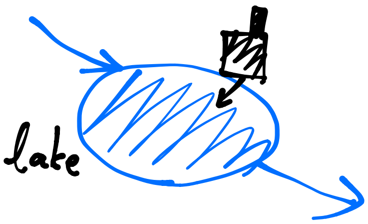
Temperature of the object changes, the rate of change is proportional to the difference of the temperature between the obj. & the room.

T — temp. of the obj. T_a — ambient temp.

$$\frac{dT}{dt} = -k(T - T_a)$$

Example: Lake, volume of water $= V\ m^3$

Assume V is constant



A factory emits mercury into the lake with a rate R kg/day

Suppose water refreshes every day by $W\ m^3/\text{day}$

How much time it takes for the water to be unpotable.

Mass of Hg at time t — $P(t)$

Take a small time period Δt

Within Δt:

Mercury in: $R\Delta t$

Polluted water out: $W\Delta t$

Density: $\frac{P(t)}{V}$

Mercury out : $\frac{P(t)}{V} W \Delta t$

Thus the change of mercury ΔP within Δt

$$\Delta P = R \Delta t - \frac{P(t)}{V} W \Delta t$$

Dividing Δt both sides:

$$\frac{\Delta P}{\Delta t} = R - \frac{P(t)}{V} W$$

Taking $\Delta t \to 0$

$$\frac{dP}{dt} = R - \frac{P}{V} W$$

EPA std: not potable when $\frac{P(t)}{V} < 0.002\ mg/L$

Assume $P(0) = 0$. We're looking for τ s.t. $\frac{P(\tau)}{V}$ arrives the threshold.

Solution of Ex. 1: assume T_a constant

$$\frac{dT}{dt} = -k(T - T_a)$$

Separating Variables:

$$\frac{dT}{T - T_a} = -k\,dt$$

Integrate both sides:

$$LHS = \int \frac{dT}{T - T_a} = \ln|T - T_a| + c$$

$$RHS = \int -k\,dt = -kt + c'$$

$$\Rightarrow \ln|T - T_a| = -kt + c$$

Exponentiate:

$$T - T_a = e^{-kt+c} = e^{-kt} e^{c} = C e^{-kt}$$

[why I can drop abs. val?]

$$T = T_a + C \cdot e^{-kt}.$$

Notice: C is arbitrary. Without further info specified there's no way to decide C.

In other words. $T(t) = T_a + Ce^{-kt}$ are solutions to the ODE for ANY C.

Such sol'ns with arbitrary constants are referred as **General Solutions.**

ODE $\Leftrightarrow$ Gen. sol'n

Solution to Example 2:

$$\frac{dP}{dt} = R - \frac{PW}{V} = R - \frac{W}{V}P$$

Separate variables:

$$\frac{dP}{R - \frac{W}{V}P} = dt.$$

Integrate both sides:

$$\text{LHS} = \int \frac{dP}{R - \frac{W}{V}P} = -\frac{V}{W} \ln\left|R - \frac{W}{V}P\right| + C$$

$$\int \frac{dP}{R - \frac{W}{V}P} = \int \frac{-\frac{V}{W}\cdot\left(-\frac{W}{V}\right) dP}{R - \frac{W}{V}P} = -\frac{V}{W}\int \frac{d\left(-\frac{W}{V}P\right)}{R - \frac{W}{V}P}$$

$$= -\frac{V}{W}\cdot \ln\left|R - \frac{W}{V}P\right| + C$$

$$\text{RHS} = t + C'$$

$$-\frac{V}{W}\ln\left|R - \frac{W}{V}P\right| = t + C$$

$$\ln\left|R-\frac{W}{V}P\right| = -\frac{W}{V}t + c$$

$$R-\frac{W}{V}P = Ce^{-\frac{W}{V}t}$$

$$P = \frac{V}{W}\left(R - Ce^{-\frac{W}{V}t}\right)$$

By assumption, $P(0) = 0$.

$$P(0) = \frac{V}{W}\left(R - C\cdot e^{0}\right) = 0$$

$$\Rightarrow C = R$$

$$P(t) = \frac{VR}{W}\left(1 - e^{-\frac{W}{V}t}\right)$$

The arbitrary constant is determined by the initial value!

Initial Value Problem (IVP) = ODE + initial value

An IVP has a unique solution

Summary: ODE (first order) ⇝ General Solution
IVP (first order) ⇝ Solution

Not all ODE can be solved in the above way!

In general, we can't expect to solve all the ODEs.

But there is a way to get some information of the sol'n

this way works for every ODE.

Geometric Interpretation

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Put $t = t_0$ to the ODE

$$\Rightarrow y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0)$$

This tells that the derivative of solution to the IVP at $t = t_0$ is known.

Example: $y' = 2y - 1, \quad y(2) = 3.$

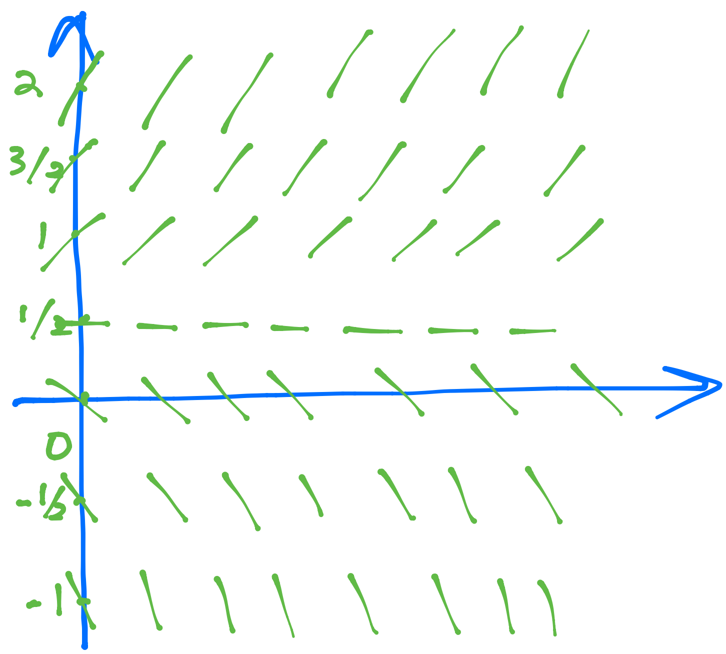
$$y'(2) = 2 \times 3 - 1 = 5.$$

Collecting all the derivatives for every pt in t-y-plane, use line elements to denote the derivative, we got a picture called direction field.

Example: $y' = 2y - 1$

RHS indep. of t.

y_0	$f(t_0, y_0)$
2	3
3/2	2
1	1
1/2	0
0	-1
-1/2	-2
-1	-3



You should know how to draw the direction field for $y' = f(y)$, (RHS not involves t)

From direction field. we'll learn at least where the solution goes: if $y' > 0$, sol'n ↗

$y' = 0$ sol'n stays

$y' < 0$ sol'n ↘

For the example $y' = 2y - 1$, if (t_0, y_0) lies above $y = \frac{1}{2}$

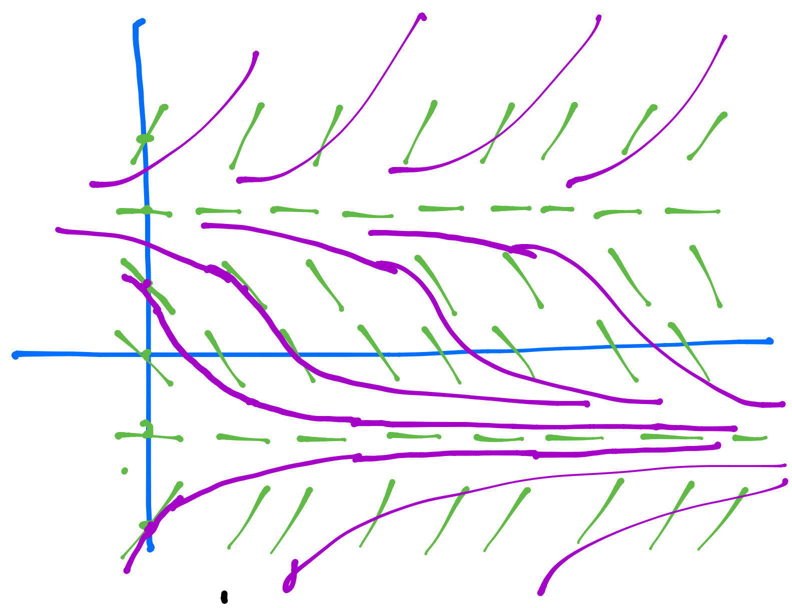
then the sol'n increases, going away from $\frac{1}{2}$

if (t_0, y_0) lies on $y = \frac{1}{2}$, then y stays $= \frac{1}{2}$

if (t_0, y_0) lies below $y = \frac{1}{2}$, then $y \searrow$, going away from $\frac{1}{2}$.

Example: $y' = (y+1)(y-2)$.

y_0	$y'(t_0)$
3	4
2	0
1	-2
0	-2
-1	0
-2	4



The actual sol'n, represented in the yt-plane, are called "integral curve" (trajectory of the solution)

Direction field in green. Integral curve in purple.

Attendance quiz: Draw the dir. field for $y' = y(y-4)$
make sure $y=4$ and $y=0$ are included.